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Original Article

On the performance of Lucas polynomials on linear and nonlinear Volterra integral equations of the second kind

Kamoh Nathaniel Mahwash[1*], Joshua Sunday[1], and Comfort Mrumun Soomiyol[2]

[1] *Department of Mathematics, University of Jos, Jos, Nigeria*

[2] *Department of Mathematics, Benue State University, Makurdi, Nigeria*

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Abstract

In this article, we consider Lucas series for $N = 2, 5, 6$ in numerical solutions of linear and nonlinear Volterra integral equations of the second kind. The series is used to transform the equation into a system of nonlinear algebraic equations, and the unknown parameters are determined. The application of this method has shown that the Lucas series is a powerful and active candidate that can be used to approximate the solution of linear and nonlinear Volterra integral equations of the second kind. The method gave a good performance, as can be seen from the four sample problems considered. The numerical results reveal computational efficiency of the method and it is also seen to be highly accurate and converge to the exact solution in some cases. Approximate and exact solutions are plotted to further confirm the accuracy of the method.

Keywords: algorithm, convergent, Volterra integral equations, second kind, linear and nonlinear

1. Introduction

In recent years, mathematical modeling of real-life problems usually results in some form of functional equations, such as algebraic equations, differential equations, integral equations, and many others. The appearance of integral equations is common in many areas of the sciences and engineering. In mathematics, an integral equation is said to be of the Volterra type if the upper limit is a variable. These equations (Volterra type) are divided into two groups referred to as the first and the second kind (Atkinson, 1997; Delves & Mohamed, 1985; William & Teukolsky, 1990). These classes of equations have gained distinction in the literature on a variety of applications, in demography as Lotka's integral equation, in studies on the risk of insolvency and on visco-elastic materials, in actuarial science through the renewal equation, and in fluid mechanics to describe the flow behavior near finite-sized boundaries. In most cases it is not possible to obtain exact solutions using analytical methods. In such cases, we need approximate solutions, and with the introduction of high-speed computers in the past few decades we have seen substantial progress in the development of approximate solutions to such problems. In literature, there are different approaches and varieties of numerical and analytical methods that are used to solve Volterra integral equations of the second kind (Al-Bugami & Al-Juaid, 2017; Altürk, 2016; Bellour & Rawashdeh, 2010; Cecilia, 2014; Kreyszig, 1979; Mandal & Bhattacharya; 2007, 2008; Mahdy, Doaa, & Lotfy, 2022; Reinkenhof, 1977).

Therefore, it is important to investigate approximate solutions of these equations. A good number of techniques have been proposed in the past, namely numerical solutions of VIE using Laguerre polynomials (Rahman, Islam, & Alam, 2012), Bernstein polynomials (Altürk, 2016; Bellour & Rawashdeh, 2010; Cecilia, 2014), or Taylor series (Wang & Wang, 2014); method of operational matrices of piecewise constant orthogonal functions (Babolian & Shamloo, 2008), Runge-Kutta method (RKM), and Block-by-Block method (Al-Bugami & Al-Juaid, 2017), Adomian and Block-by-Block methods (EL-Kalla & AL-Bugami, 2011), two-step collocation (2-SC) method (Torabi & Shahmorad, 2019), multistep collocation method (Jingjun, Teng, & Xu, 2019), the well-

*Corresponding author
Email address: mahwash1477@gmail.com

known Galerkin method using Laguerre and Hermite polynomials as trial functions (Rahman *et al.*, 2012), Lucas polynomials for approximating solution of Cauchy integral equation (Mahdy & Mohamed, 2022), computing second-type mixed integral equations with singular kernels (Mahdy, *et al.*, 2023) and (Kamoh, Ali, & Dang, 2022*)*. The matrix approach method has been developed (Kamoh, Kumleng, & Sunday, 2020).

Recently, a variety of specialized methods (Abeer, Abdou & Mahdy, 2023; Kamoh, *et al.*, 2020, 2022; Khaled, *et al.*, 2021; Mahdy, Abdou & Mohamed, 2024; Mahdy, *et al.*, 2023; Mahy, 2023; Rahman *et al.*, 2012) have been reported in literature. These methods have generated impressive and accurate numerical results for the problems considered in examples or experiments.

1.1 Lucas polynomial

To define the Lucas polynomial, we need to first understand what Lucas numbers are. The Lucas numbers are a sequence of integers in which each number is the sum of its two immediately preceding numbers, similar to the Fibonacci sequence. The Lucas numbers are denoted by $L(n)$ and are defined as:

$$L(0) = 2$$
$$L(1) = 1,$$
$$L(n) = L(n-1) + L(n-2) \; for \; n \geq 2$$

The Lucas polynomials use the same recurrence with different starting values

$$l_n(x) = \begin{cases} 2 & if \; n = 0 \\ x & if \; n = 1 \\ xL_{n-1}(x) + L_{n-2}(x) & if \; n \geq 2 \end{cases}$$

or

$$l_n(x) = 2^{-n}\left(x - \sqrt{x}^2 + 4\right)^n\left(x + \sqrt{x}^2 + 4\right)^n$$

The first few Lucas polynomials are given below.

$$l_0(x) = 2$$
$$l_1(x) = x$$
$$l_2(x) = x^2 + 2$$
$$l_3(x) = x^3 + 3x$$
$$l_4(x) = x^4 + 4x^2 + 2$$
$$l_5(x) = x^5 + 5x^3 + 5$$
$$l_6(x) = x^6 + 6x^4 + 9x^2 + 2$$

The Binet and power form representations of the Lucas polynomials can be found in (Abd-Elhameed & Youssri, 2016, 2017).

1.2 The existence and uniqueness theorem (Volterra theorem)

Some integral equations have solutions and some others have no solutions, or they can have an infinite number of solutions. The following theorems state the existence and uniqueness of solution for the Volterra integral equations of the second kind.

Theorem 1.1 Assume that the kernel $K(x,t)$ of the linear Volterra integral equations of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)u(t)\,dt, \qquad x \in I = [0,T],$$

is continuous on $D := \{(x,t): 0 \leq t \leq x \leq T\}$. Then for any function $f(x)$ that is continuous on I (i.e., $f \in C(I)$), the Volterra integral equation possesses a unique solution $u \in C(I)$. This solution can be written in the form

$$u(x) = f(x) + \int_0^x R(x,t)f(t)\,dt, \;\; x \in I$$

for some $R \in C(D)$. The function $R = R(x,t)$ is called the resolving kernel of the given kernel $K(x,t)$, (Brunner, 2010).

Theorem 1.2 If we define the integral operator $K: C(I) \to C(I)$ by

$$(Kf)(x) := \int_0^x R(x,t)f(t)\,dt, \;\; x \in I$$

then, the Volterra integral equation in operator form is given by

$$u = f + Vu \;\; \text{Or} \;\; (I - V)u = f$$

(where I denotes the identity operator, and the classical Volterra operator $V: C(I) \to C(I)$ is defined by $(Vu)(x) := \int_0^x R(x,t)f(t)\,dt,\ \ x \in I$, with $K \in C(D)$), and we have the following relationship

$$(I - V)u = f \Rightarrow u = (I + K)f$$

By Theorem 1.1 this implies that the inverse $(I - V)^{-1}$ always exists, and hence (by uniqueness of $R(x,t)$) $(I - V)^{-1} = I + K$, (Brunner, 2010).

This paper is structured as follows. In Section 2, the considered class of equations is introduced and the technique is eloquently discussed. Section 3 deals with the illustration of the numerical method through some problems, and the outcomes are compared with other existing methods. Finally, in Section 4 some concluding remarks are discussed.

2. Context and Method of Solution

The purpose of this paper is to demonstrate recent results on the Volterra integral equation of the second kind (VIESK) of the form;

$$g(x)y(x) = f(x) + \rho \int_a^x k(x,t)y(t)dt\,, -\infty < a \le x \le b < \infty \quad (1.0)$$

where ρ is real constant. The function $f(x)$ and the kernel $k(x,t)$ are known. The solution $y(x)$ of (1.0) is to be sought for $g(x) = 1$.

In this article, we propose a polynomial based method similar to (Kamoh, *et al.*, 2020) and collocation approach to numerically solve (1.0), demonstrating its desirable properties. The collocation method provides an approximation over the entire integration interval to the solution of the equation, which is revealed to be quite useful in a variable-step size implementation; indeed, it is easy to recover the missing past values when the step size is changed by evaluating the collocation polynomial. Other good properties of this approach are its high order of convergence, strong stability properties, and flexibility. As a matter of fact, if some information is known on the behavior of the exact solution, then it is possible to choose collocation functions in order to better follow such behavior and this gives rise to mixed collocation methods (Cardone, Conte, D'ambrosio, & Paternoster, 2018).

The novelty of the present approach is, to the best of our knowledge, that no method similar to the proposed method has been discussed in any literature to date. It is our strong belief that many will find the proposed method appealing, and it is an improvement to existing methods for the numerical solution of Volterra integral equations of the second kind.

The proposed technique to solve equation (1.0) is based on the finite Lucas series of the form

$$y_N(x) = \sum_{i=0}^{N} a_i l_i(x) \quad (2.0)$$

Here $a_i,\ i = 0,1,\ldots,N$ are unknown coefficients and $l_i(x), i = 0,1,\ldots,N;$ are the Lucas polynomials (Lucas, 1878, Liu, 2013).

Equation (1.0) with $g(x) = 1 = \rho$ takes the form;

$$y(x) = f(x) + \int_a^x k(x,t)y(t)dt \quad (3.0)$$

We assume that (2.0) is an approximate solution of (3.0), where $l_i(x)$ is the Lucas polynomial of degree i defined in equation (2.0) and $a_i's$ are the unknown parameters to be determined. Substituting (2.0) into (3.0) gives

$$\sum_{i=0}^{N} a_i l_i(x) - \int_a^x k(x,t) \sum_{i=0}^{N} a_i l_i(t)\,dt = f(x) \quad (4.0)$$

Expanding the integral in (4.0), we get the result

$$\sum_{i=0}^{N} a_i l_i(x) - \left(a_0 \int_a^x k(x,t) l_0(t)dt + \cdots + a_N \int_a^x k(x,t) l_n(t)dt\right) = f(x) \quad (5.0)$$

Evaluating (5.0) at the points

$$x = x_i = \frac{i}{N}, i = 0,1,\ldots,N, \quad x\epsilon[a, x_N]$$

gives an $(N+1)$ by $(N+1)$ system of linear equations, which can be solved (using Gaussian elimination) for $a_0, a_1, a_2, \ldots, a_N$ and substituting the calculated values of $a_0,\ a_1,\ a_2, \ldots, a_N$ into (2.0), so an approximate solution for (3.0) is obtained.

3. Application of the Method

In (Kamoh, *et al.*, 2022), the researchers used a very simple and efficient Galerkin weighted residual method with Hermite polynomials as trial function to solve Volterra integral equations of the first kind, while (Zarnan, 2016) used trapezoidal rule to solve Volterra integral equations of the second kind. The present work suggests that the proposed method is comparatively simpler to apply than most existing methods. Four numerical examples are solved in order to further illustrate the simplicity and applicability of this method. These test problems were previously solved by (Al-Bugami & Al-Juaid, 2017; Majeed & Jabar, 2014; Zarnan, 2016). All calculations are performed with Lucas series for $N = 2, 5, 6$ using Scientific Workplace 5.5 software. The detailed steps are shown below.

Problem 3.1 Consider the non-linear Volterra integral equation of the second kind (Al-Bugami & Al-Juaid, 2017).

$$y(x) = x + \frac{1}{5}x^5 - \int_0^x t\big(y(t)\big)^3 dt, \quad 0 \le x \le 1$$

for which the exact solution is $y(x) = x$. Applying the present technique with $N = 2$, and collocating (5.0) at $x_i = \frac{i}{2}, i = 0,1,2$ and solving the resulting system of equations, we obtain

$$[a_0 = 0, a_1 = 1.0, a_2 = 0]$$

Substituting these approximate values into (2.0), we get the approximate solution to the problem as

$$y_N(x) = x$$

The approximate solution is the same as the exact solution showing the accuracy of the method. Numerical results by (Al-Bugami & Al-Juaid, 2017) for $N = 50$ are compared with the present method for $N = 2$ in Table 1

Problem 3.2 Here we solve equation (3.0) with $k(x,t) = e^{-(x-t)}$, $f(x) = 1$ and the exact solution is $y(x) = x + 1$. Applying the present technique with $N = 5$, and collocating (5.0) at $x_i = \frac{i}{5}, i = 0,1,2,\dots,5$ and solving the resulting system of equations, we obtain

$$[a_0 = \frac{1}{2}, a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0]$$

Substituting these values into (2.0), we get an approximate solution to the problem

$$y_N(x) = x + 1$$

The approximate solution is the same as the exact solution affirming further the accuracy of the present method. Numerical results by (Majeed & Jabar, 2014) for $N = 10$ are compared with the present method with $N = 5$ in Table 2.

Problem 3.3 Consider the linear Volterra integral equation of the second kind solved by (Zarnan, 2016).

$$y(x) - \int_0^x (t-x)y(t)y(t)dt = x, \qquad 0 \le x \le 1$$

where $k(x,t) = (t-x)$ and $f(x) = x$ with exact solution given by $y(x) = sinx$. Applying the present technique with $N = 6$ and collocating at $x_i = \frac{i}{6}, i = 0,1,2,\dots,6$ and solving for the unknown parameters from the resulting system of equations, we obtain

$$[a_0 = 4.79765385676 \times 10^{-3}, a_1 = 1.59162091428, a_2 = -7.48542224102 \times 10^{-3},$$
$$a_3 = -0.212588646381, a_4 = 3.34903018189 \times 10^{-3}, a_5 = 9.22955565935 \times 10^{-3}$$
$$a_6 = -6.61261797629 \times 10^{-4}]$$

Substituting these values into (2.0), we get the approximate solution

$$y_N(x) = -6.61261797629 \times 10^{-4}\, x^6 + 9.22955565935 \times 10^{-3}\, x^5 - 6.18540603889 \times 10^{-4}\, x^4$$
$$- 0.166440868084\, x^3 - 4.06576921387 \times 10^{-5}\, x^2 + 1.00000275343\, x$$

The approximate results are compared to the exact results in Table 3, and the approximate solution is plotted against the exact solution to further confirm the accuracy of the present method in Figure 1.

Problem 3.4 Consider the nonlinear Volterra integral equation solved by (Al-Bugami & Al-Juaid, 2017).

$$y(x) = sinx + \frac{x(1-cos2x)}{16} + \frac{x^2(x-sin2x)}{8} - \int_0^x \frac{tx}{2}\big(y(t)\big)^2 dt, \qquad 0 \le x \le 1$$

where $k(x,t) = \frac{tx}{2}$ and $f(x) = sinx + \frac{x(1-cos2x)}{16} + \frac{x^2(x-sin2x)}{8}$ with exact solution given by $y(x) = sinx$. Applying the present technique with $N = 2$ and collocating at $x_i = \frac{i}{2}, i = 0,1,2$ and solving the resulting system of equations, we obtain

$$[a_0 = 0.232670254\,801, a_1 = 1.075039216\,34, a_2 = -0.232670254\,801]$$

Substituting these approximate values into (2.0), we obtain the approximate solution to the problem as

$$y_N(x) = 1.075039216\,34x - 0.232670254\,801\, x^2$$

The approximate solution by (Al-Bugami & Al-Juaid, 2017) for $N = 80$ is compared with the present method with $N = 2$ in Table 4. Also, the approximate and exact solutions were plotted to further confirm the accuracy of the present method, in Figure 2.

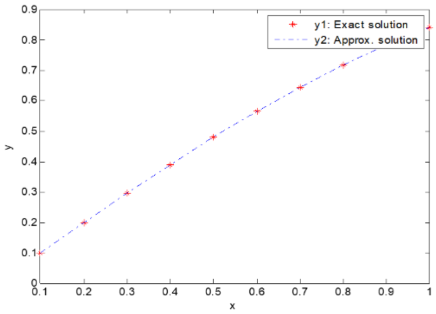

Figure 1. Comparing exact and approximate solutions to problem 3.3 for N=6

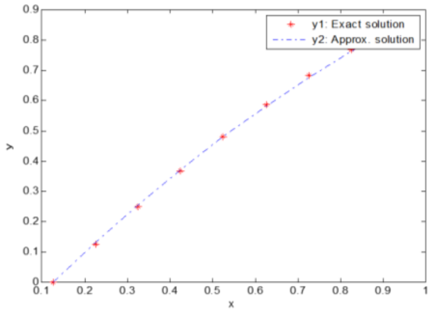

Figure 2. Comparing exact and approximate solutions to problem 3.4 for N=2

Table 1. Computed exact and approximate solutions and absolute errors for $N = 2$

x	Exact solution	Approx solution of proposed method	Absolute error of proposed method	Approx solution by (Al-Bugami & Al-Juaid, 2017) using Runge Kutta method for $N = 50$	Absolute error by (Al-Bugami & Al-Juaid, 2017) using Runge Kutta method $N = 50$
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1	0.1	0.0	0.0997267933	2.73×10^{-4}
0.2	0.2	0.2	0.0	0.1992790511	7.21×10^{-4}
0.3	0.3	0.3	0.0	0.2942789348	5.72×10^{-3}
0.4	0.4	0.4	0.0	0.3903370122	9.66×10^{-3}
0.5	0.5	0.5	0.0	0.4903037547	9.70×10^{-3}
0.6	0.6	0.6	0.0	0.5703260241	2.97×10^{-2}
0.7	0.7	0.7	0.0	0.6615191696	3.85×10^{-2}
0.8	0.8	0.8	0.0	0.7396378531	6.04×10^{-2}
0.9	0.9	0.9	0.0	0.8344427662	6.56×10^{-2}
1.0	1.0	1.0	0.0	0.9055801198	9.44×10^{-2}

Table 2. Computed exact and approximate solutions and absolute errors for $N = 5$

x	Exact solution	Approximate solution of proposed method for $N = 5$	Absolute error of proposed method for $N = 5$	Absolute error by (Majeed & Jabar, 2014) for $N = 10$
0.0	1.0	1.0	0.0	0.0
0.2	1.2	1.2	0.0	1.5019×10^{-10}
0.4	1.4	1.4	0.0	3.0462×10^{-10}
0.6	1.6	1.6	0.0	4.6328×10^{-10}
0.8	1.8	1.8	0.0	6.2616×10^{-10}
1.0	2.0	2.0	0.0	7.9328×10^{-10}

Table 3. Computed exact and approximate solutions and absolute errors for $N = 6$

x	Exact solution	Approximate solution of proposed method for $N = 6$	Absolute error of proposed method for $N = 6$	Absolute error by (Zarma, 2016) for $N = 10$
0.0	0.0	0.0	0.0	0.0
0.1	0.0998334166468	0.0998334166468	4.10×10^{-8}	1.67×10^{-4}
0.2	0.198669330795	0.198669330795	1.19×10^{-8}	3.31×10^{-4}
0.3	0.295520206661	0.295520206661	7.68×10^{-9}	4.90×10^{-4}
0.4	0.389418342309	0.389418342309	5.76×10^{-9}	6.42×10^{-4}
0.5	0.479425538604	0.479425538608	2.21×10^{-9}	7.84×10^{-4}
0.6	0.564642473395	0.564642473429	1.01×10^{-8}	9.14×10^{-4}
0.7	0.644217687238	0.644217687453	3.17×10^{-9}	1.03×10^{-3}
0.8	0.717356090900	0.717356091939	7.22×10^{-9}	1.13×10^{-3}
0.9	0.783326909627	0.783326913749	4.29×10^{-8}	1.21×10^{-3}
1.0	0.841470984808	0.841470998816	3.89×10^{-9}	1.28×10^{-3}

Table 4. Computed exact and approximate solutions and absolute errors for $N = 2$

x	Exact solution	Approx solution of proposed method $N = 2$	Absolute error of proposed method $N = 2$	Approx solution by (Al-Bugami1 & Al-Juaid, 2017) using Runge Kutta method for $N = 80$	Absolute error by (Al-Bugami & Al-Juaid, 2017) using Runge Kutta method for $N = 80$
0.000	0.0	0.0	0.0	0.0	0.0
0.125	0.1246747334	0.130744429 311	6.07×10^{-3}	0.1246835292	8.80×10^{-6}
0.250	0.2474039593	0.254217913 160	6.81×10^{-3}	0.2476985837	2.95×10^{-4}
0.375	0.3662725291	0.370420451 546	4.15×10^{-3}	0.3676282777	1.36×10^{-3}
0.500	0.4794255385	0.479352044 470	7.35×10^{-5}	0.4830800698	3.65×10^{-3}
0.625	0.5850972724	0.581012691 931	4.08×10^{-3}	0.5926912852	7.59×10^{-3}
0.750	0.6816387600	0.675402393 929	6.24×10^{-3}	0.6951347139	1.35×10^{-2}
0.875	0.7675434022	0.762521150 465	5.02×10^{-3}	0.7891248129	2.16×10^{-2}
1.000	0.8414709848	0.842368961 539	8.98×10^{-4}	0.8734243449	3.20×10^{-2}

5. Conclusions and Discussion

The advantage of the present work is that the proposed method is comparatively simpler to apply than most existing methods, whereas the numerical results and graphical illustrations depict the accuracy and superiority of the present method. The main attraction of the present method is displayed by the comparative study. The superior results for different input values testify to novelty of the present work. The applications of this method have shown that the Lucas series is a powerful and active candidate for approximating solutions to linear and nonlinear Volterra integral equations of the second kind. The method gave a good approximate solution in the four sample problems considered and the numerical results revealed that the method is computationally efficient. Tables 1, 2, 3 and 4 present the absolute errors, whereas the plots in Figures 1 and 2 confirm that as the order of the Lucas series increases, the approximate solution converges to the exact solution. The idea presented in this work suggests the possibility of replicating similar arguments applied to integro-differential equations of the Fredholm or Volterra types. Work is currently ongoing in this regard.

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